PART – I: MATHEMATICS

SECTION - I(Single Correct Choice Type)

This is section contains **8 multiple choice questions**. Each question has four choices A), B), C) and D) out of which **ONLY ONE** is correct.

29. The number of 3 × 3 matrices A whose entries are either 0 or 1 and for which the system

$A\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ has exactly two distinctions, is

A) 0 B) $2^9 - 1$ C) 168 D) 2

Ans: (A)

Sol. $a_1x + b_1y + c_1z = 1$

$a_2x + b_2y + c_2z = 0$

$a_3x + b_3y + c_3z = 0$

No three places can meet at two distinct points. So number of matrices is 0.

30. The value of $\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\ln(1+t)}{t^4+4}dt\ \ is$

A) 0 B) $\frac{1}{12}$ C) $\frac{1}{24}$ D) $\frac{1}{64}$

Ans: (B)

Sol. $\lim_{x\to 0}\frac{x\ln(1+x)}{(x^4+4)3x^2} = \lim_{x\to 0}\frac{1}{4}\times\frac{1}{3} = \frac{1}{12}$

31. Let p and q be real numbers such that $p \neq 0, p^3 \neq q\ and\ p^3 \neq -q$. If $\alpha\ and\ \beta$ are non zero complex numbers satisfying $\alpha + \beta = -p\ and\ \alpha^3 + \beta^3 = q$, than a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is

A) $(p^3+q)x^2 - (p^3+2q)x + (p^3+q) = 0$ B) $(p^3+q)x^2 - (p^3-2q)x + (p^3+q) = 0$

C) $(p^3-q)x^2 - (5p^3-2q)x + (p^3-q) = 0$ D) $(p^3-q)x^2 - (p^3+2q)x + (p^3-q) = 0$

Ans: (B)

Sol. Product = 1

Sum = $\frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

Since = $\alpha^3+\beta^3 = q \quad \Rightarrow \quad -p(\alpha^2+\beta^2-\alpha\beta) = q$

$((\alpha+\beta)^2 - 3\alpha\beta) = -\frac{q}{p} \quad \Rightarrow \quad p^2 + \frac{q}{p} = 3\alpha\beta$

Hence sum = $\frac{\left\{p^2-\frac{2}{3}\left(\frac{p^3+q}{p}\right)\right\}3p}{(p^3+q)} = \frac{p^3-2q}{p^3+q}$

So the equation is $x^2 + \left(\frac{p^3-2q}{p^3+q}\right)x + 1 = 0$

$\Rightarrow (p^3+q)x^2 - (p^3-2q)x + (p^3+q) = 0$

32. Equation of the plane containing the straight line $\frac{x}{2}=\frac{y}{x}=\frac{z}{4}$ and perpendicular the plane containing the straight line $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$ is

A) $x+2y-2z=0$ B) $3x+2y-2z=0$ C) $x-2y+2z=0$ D) $5x+2y-4z=0.$

Ans: (C)

Sol. Direction ratio of normal to plane containing the straight line

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{vmatrix} = 8\hat{\imath}-\hat{\jmath}-10\hat{k}$$

Required plane $\begin{vmatrix} x-0 & y-0 & z-0 \\ 2 & 3 & 4 \\ 8 & -1 & -10 \end{vmatrix}=0 \Rightarrow -26x+52y-26z=0 \Rightarrow x-2y+z=0.$

33. If the angles A, B and C of a triangle are in an arithmetic progression and if a, b and c denotes the length of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2c+\frac{c}{a}\sin 2A\ is$

A) $\frac{1}{2}$ B) $\frac{\sqrt{3}}{2}$ C) 1 D) $\sqrt{3}$

Ans:(D)

Sol. $\frac{a}{c}\sin 2C+\frac{c}{a}\sin 2A=\frac{2}{2R}(a\cos C+c\cos A)=\frac{b}{R}=2\sin B=2\sin 60^0=\sqrt{3}.$

34. Let f, g and h be real – valued functions defined on the interval [0, 1] by $f(x)=e^{x^2}+e^{-x^2}$, $g(x)=xe^{x^2}+e^{-x^2}\ and\ h(x)=x^2e^{x^2}+e^{-x^2}$. if a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then

A) $a=b\ and\ c\neq b$ B) $a=c\ and\ a\neq b$ C) $a\neq b\ and\ c\neq b$ D) $a=b=c$

Ans: (D)

Sol. Clearly f(x) = $e^{x^2}+e^{-x^2}$

f'(x) = 2x $\left(e^{x^2}-e^{-x^2}\right)\geq 0\ increasing\ \Rightarrow f_{max}=f(1)=e+\frac{1}{e}$

g(x) = $xe^{x^2}+e^{-x^2}\Rightarrow g(x)=e^{x^2}+2x^2e^{x^2}-2x^2e^{-x^2}>0\ increasing$

$$\Rightarrow g_{max}=g(1)=e+\frac{1}{e}$$

h(x) = $x^2e^{x^2}+e^{-x^2}\Rightarrow h'^{(x)}=2xe^{x^2}+2x^3e^{x^2}-2x^2e^{-x^2}=2x\left(e^{x^2}+x^2e^{x^2}-e^{-x^2}\right)>0$

$\Rightarrow h_{max}=h(1)=e+\frac{1}{e}.$

So a = b = c.

35. Let ω be a complex cube root of unity with ω ≠ 1. A fair die is thrown three times. If $r_1, r_2\ and\ r_3$ are the numbers obtained on the die, then the probability that $\omega^{r_1}+\omega^{r_2}+\omega^{r_3}=0\ is$

A) $\frac{1}{18}$ B) $\frac{1}{9}$ C) $\frac{2}{9}$ D) $\frac{1}{36}$

Ans: (C)

Sol. $\omega^{r_1}+\omega^{r_2}+\omega^{r_3}=0; r_1, r_2, r_3\ are\ to\ be\ selected\ form\{1,2,3,4,5,6\}$

As we know that $1 + \omega + \omega^2 = 0$

$\therefore$ From r_1, r_2, r_3, one has remainder 1, other has remainder 2 and third has remainder 0 when divided by 3 we have to select r_1, r_2, r_3 *from* $(1,4)$ *or* $(2,5)$ *or* $(3,6)$ *which can be done in* $C_1^2 \times C_1^2 \times C_1^2$ *ways value of* r_1, r_2, r_3 *can be interchanged in* 31 *ways*.

Required probability = $\frac{(C_1^2 \times C_1^2 \times C_1^2) \times 3!}{6 \times 6 \times 6} = \frac{2}{9}$.

36. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{\imath} - \hat{\jmath}, 4\hat{\imath}, 3\hat{\imath} + 3\hat{\jmath}$ *and* $-3\hat{\imath} + 2\hat{\jmath}$ respectively. The quadrilateral PQRS must be a

A) Parallelogram, which is neither a rhombus nor a rectangle

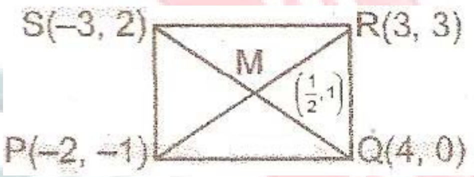
B) Square

C) Rectangle, but not a square

D) Rhombus, but not a square

Ans: (A)

Sol. PQ = $\sqrt{36 + 1} = \sqrt{37} = RS,\ PQ \neq PS$



PS = $\sqrt{1 + 9} = \sqrt{10} = QR$

Slope of PQ = $\frac{1}{6}$, *slope of PS* $= -3$

PQ is not $\perp$ *to PS*

So it is parallelogram, which is neither a rhombus nor a rectangle.

SECTION –II(Multiple Correct Choice Type)

This section contains 5 multiple choice question. Each question has four choices A), B),C),D) out of which ONE OR MORE may be correct.

37. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg(w) denotes the principal argument of a nonzero complex number w, then

A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$ B) $Arg\ (z - z_1) = Arg\ (z - z_2)$

C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$ D) $Arg\ (z - z_1) = Arg\ (z_2 - z_1)$

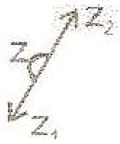
Ans: (A),(C),(D)

Sol. (A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

AB + BC = AC

(B) $Arg\ (z - z_1) = Arg\ (z - z_2) = \pi$

(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z_1} \\ z_2 - z_1 & \bar{z_2} - \bar{z_1} \end{vmatrix} = 0$



(D) $Arg\,(z - z_1) = Arg\,(z_2 - z_1)$

38. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope o the joining A and B can be

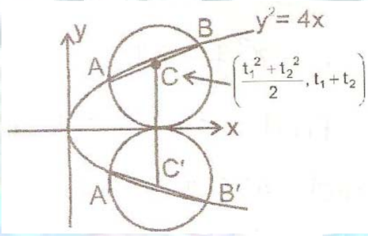
A) $-\frac{1}{r}$ B) $\frac{1}{r}$ C) $\frac{2}{r}$ D) $-\frac{2}{r}$

Ans: (C),(D)

Sol. $A(t_1^2, 2t_1), B(t_2^2, 2t_2)$,

Centre of circle $\left(\frac{t_1^2+t_2^2}{2}, t_1 + t_2\right)$

$\Rightarrow |t_1 + t_2| = r, slope\ of\ AB = \frac{2}{t_1+t_2} = \pm\frac{2}{r}$



39. Let f be a real – valued function defined on the interval (0, ∞) by $f(x) = \ln x + \int_0^x \sqrt{1 + \sin t}\, dt$. Then which of the following statement(s) is (are) true

A) f'' (x) exist for all $x \epsilon (0, \infty)$

b) f'(x) exist for all $x \epsilon (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$

C) there exists α > 1 such that $|f'(x)| < |f(x)|$ for all $x \epsilon (\alpha, \infty)$

D) there exists β > 0 such that $|f(x)| + |f'(x)| \leq \beta$ for all $x \epsilon (\alpha, \infty)$

Ans: (B),(C)

Sol. $f(x) = \ln x + \int_0^2 \sqrt{1 + \sin t}\, dt$

$f'(x) = \frac{1}{x} + \sqrt{1 + sinx}$

$f''(x) = -\frac{1}{x^2} + \frac{\cos x}{2\sqrt{1+sinx}}$

$(A) f''\ is\ defined\ for\ x = -\frac{\pi}{2} + 2n\pi, n \in I$

So (A) is wrong

(B) f'(x) always exist for x > 0

(C) $|f'| < |f|$ since f' > 0 and f > 0 f' < f
$\frac{1}{x} + \sqrt{1+sinx} < \ln x + \int_0^x \sqrt{1+sinx}\, dx$
LHS is bounded *RHS is increasing with range* ∞
So there exist some α beyond which RHS is greater than LHS

(D) $|f| + |f'| \le b$ is wrong as f is MI & its range is not bounded while is infinite.

40. The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$ is (are)

A) $\frac{22}{7} - \pi$ B) $\frac{2}{105}$ C) 0 D) $\frac{71}{15} - \frac{3\pi}{2}$

Ans: (A)

Sol. $\int_0^1 \frac{x^4(1-x)^4}{1+x^2} = \int_0^1 \frac{x^4[(1+x^2)-2x]^2}{1+x^2} = \int_0^1 \frac{x^4[(1+x^2)^2-4x(1+x^2)+4x^2]}{1+x^2}$

$= \int_0^1 x^4\left[(1+x^2) - 4x + \frac{4x^2}{1+x^2}\right] dx = \int \left[x^6 + x^4 - 4x^5 + \frac{4x^6}{1+x^2}\right] dx$

Now on polynomial division of x^6 *by* $1 + x^2$, *we obtain*

$\int \left[x^6 + x^4 - 4x^5 + 4\left[(x^4 - x^2 + 1) - \frac{1}{1+x^2}\right]\right] dx = \left[(x^6 - 4x^5 + 5x^4 - 4x^2 + 4) - \frac{4}{1+x^2}\right] dx$

$= \left[\frac{x^7}{7} - \frac{4x^6}{6} + \frac{5.x^5}{5} - \frac{4x^3}{3} + 4x\right]_0^1 - 4[\tan^{-x} x]_0^1 = \left(\frac{1}{7} - \frac{4}{6} + 1 - \frac{4}{3} + 4\right) - 4\left(\frac{\pi}{4}\right) = \left[\frac{1}{7} - \frac{12}{6} + 5\right] - z$

$= \left(\frac{1}{7} + 3\right) - \pi = \frac{22}{7} - \pi$

41. Let ABC be a rectangle such that $\angle ACB = \frac{\pi}{6}$ and a, b and c denote the length of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1, b = x^2 - 1$ *and* $c = 2x + 1$ is (are)

A) $-(2+\sqrt{3})$ B) $1+\sqrt{3}$ C) $2+\sqrt{3}$ D) $4\sqrt{3}$

Ans:(A, B)

Sol. $\cos\frac{\pi}{6} = \frac{(x^2-1)^2+(x^2+x+1)^2-(2x+1)^2}{2(x^2+x+1)(x^2-1)}$

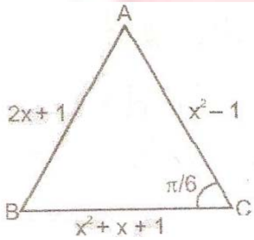
$\frac{\sqrt{3}}{2} = \frac{(x^2-1)^2+(x^2+3x+2)-(x^2-x)}{2(x^2+x+1)(x^2-1)}$ then $\frac{\sqrt{3}}{2} = \frac{(x^2-1)^2+(x+1)(x+2)x(x-1)}{2(x^2+x+1)(x^2-1)}$

$\Rightarrow \sqrt{3} = \frac{x^2-1+x(x+2)}{x^2+x+1} \Rightarrow \sqrt{3}(x^2+x+1) = 2x^2 + 2x - 1$

$\Rightarrow (\sqrt{3}-2)x^2 + (\sqrt{3}-2)x + (\sqrt{3}+1) = 0$

on solving

$\Rightarrow x^2 + x - (3\sqrt{3}+5) = 0\ x = \sqrt{3}+1, -(2+\sqrt{3})$



SECTION – III (Paragraph Type)

This Section contains 2 paragraphs. Based upon the first paragraph 3 multiple choice questions and based upon the second paragraph 2 multiple choice questions have to be answered. Each of these questions has four choices A), B), C) and D) out of which ONELY ONE is correct.

Paragraph for Questions 42 to 44

Let p be an odd prime number and T_P be the following set of 2 × 2 matrices:

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, 2, \ldots, p-1\} \right\}$$

42. The number of A in T_p such that a is either symmetric or skew – symmetric or both, and det (A) divisible by p is

A) $(p-1)^2$ B) $2(p-1)$ C) $(p-1)^2+1$ D) $2p-1$

Ans(D)

Sol. $A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$ $where\ a, b, c \in \{0, 1, 2, \ldots, p-1\}$

Case – I A is symmetric matrix $\Rightarrow b = c$

$\Rightarrow \det(A) = a^2 - b^2\ is\ divisible\ by\ p$

$\Rightarrow (a-b)(a+b) = is\ divisible\ by\ p$

$(a) a - b\ is\ divisible\ by\ p\ if\ a = b, then\ 'p'\ cases\ are\ possible$

$(b) a + b\ is\ divisible\ by\ p\ if\ a + b = p, then \dfrac{(p-1)}{2} \times 2 = (p-1) cases\ are\ possible$

Case – II

A is skew symmetric matrix

If a = 0, b + c = 0, then det (A) = b^2

$\Rightarrow b^2\ can\ ever\ be\ divisible\ by\ p.\ So\ No\ case\ is\ possible$

Total number of A is possible = 2p - 1

43. The number of A in T_p such that the trace of A is not divisible by p but det (A) divisible by p is

A) $(p-1)(p^2-p+1)$ B) $p^3-(p-1)^2$ C) $(p-1)^2$ D) $(p-1)(p^2-2)$

Ans: (C)

Sol. $a^2 - bc \div p$

A can be chosen in p – 1 ways $a \neq 0)$

Let a be 4 & p = 5

So a^2 = 16 and hence bc should be chosen such that $a^2 - bc \div p$

Now b can be chosen in p – 1 ways and c in only (one be is chosen)

Explanation: if b = 1 $\Rightarrow c = 1$

b = 2 $\Rightarrow c = 3$

b = 3 $\Rightarrow c = 2$

b = 4 $\Rightarrow c = 4$

Hence a can be chosen in p – 1 ways And then b can be chosen in p – 1 ways

So $(p-1)^2$.

44. The number of A in T_p such that det (A) is not divisible by p is

A) $2p^2$ B) $p^3 - 5p$ C) $(p^3 - 3p$ D) $p^3 - p^2$

Ans: (D)

Sol. As = Total cases $-a \neq 0\ and\ |A|\ is\ divisible\ by\ p) - (a = 0\ and\ |A|\ is\ divisible\ by\ p)$

$= p^3 - (p-1)^2 - (2p-1) = p^3 - p^2$

$-bc\ \div p\ since\ b\ \&\ c\ both\ we\ coprime\ to\ p$

$\Rightarrow$ one of them must be zero.

$if\ b = 0 \Rightarrow c\ can\ be\ chosen\ in\ \{0, 1, \ldots, p-1\}$

$if\ c = 0 \Rightarrow b\ can\ be\ chosen\ in\ \{0, 1, \ldots, p-1\}$

Paragraph for Questions 45 to 46

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$intersect at the point A and B.

45. Equation of a common tangent with positive slope to the circle a as well as to the hyper bola is

A) 2x -$\sqrt{5}y - 20 = 0$ B) 2x -$\sqrt{5}y + 4$=0 C) 3x-4y+8 = 0 D) 4x -3y +4 =0

Ans:(B)

Sol. Let equation of tangent to ellipse

$\frac{\sec\theta}{3}x - \frac{\tan\theta}{2}y = 1$

$2\sec\theta\, x - 3\tan\theta\, y = 6$

It is also tangent to circle $x^2 + y^2 - 8x = 0$

$\Rightarrow \frac{|8\sec\theta - 6|}{\sqrt{4\sec^2\theta + 9\tan^2\theta}} = 4$

$(8\sec\theta - 6)^2 = 16(13\sec^2\theta - 9)$

$\Rightarrow 12\sec^2\theta + 8\sec\theta - 15 = 0$

$\Rightarrow \sec\theta = \frac{5}{6}\ and\ -\frac{3}{2}\ but\ \sec \neq \frac{5}{6}$

$\Rightarrow \sec\theta = -\frac{3}{2} \quad \Rightarrow \tan\theta = \frac{\sqrt{3}}{2} \qquad \therefore slope\ is\ positive$

Equation of tangent = 2x $-\sqrt{5}y + 4 = 0$

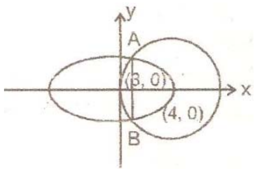
46. Equation of the circle with AB as its diameter is

A) $x^2 + y^2 - 12x + 24 = 0$ B)$x^2 + y^2 + 12x + 24 = 0$

C) $x^2 + y^2$-24x -12=0 D) $x^2 + y^2 - 24x - 12 = 0$

Ans: (A)

Sol. $x^2 + y^2 - 8x = 0$



$\frac{x^2}{9} - \frac{y^2}{4} = 1 \Rightarrow 4x^2 - 9y^2 = 36$

$\Rightarrow 4x^2 - 9(8x - x^2) = 36$

$13x^2 - 72x - 36 = 0$

$13x^2 - 78x + 6x - 36 = 0$
$(13x+6)(x-6) = 0$
$\Rightarrow x = -\frac{6}{13}$ *and* $x = 6$
but $x > 0 \Rightarrow x = 6$
$\Rightarrow A(6, \sqrt{2})$ *and* $B\ (6, -\sqrt{2})$
⇒ equation of circle with AB as a diameter $x^2 + y^2 - 12x + 24 = 0$.

SECTION –IV (Integer Type)

This section contains **TEN questions** the answer to each question is a **single – digit integer**, ranging from 0 to 9. The correct digit below the question number in the ORS is to be bubbled.

47. Let ω *be the complex number* $cos\frac{2\pi}{3} + i\ sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to

Ans: 1

Sol. $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} z & \omega & \omega^2 \\ z & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0 \quad \Rightarrow z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

z = 0

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ 0 & z+\omega^2-\omega & 1-\omega^2 \\ 0 & 1-\omega & z+\omega-\omega^2 \end{vmatrix} = 0$$

$(z+\omega^2-\omega)(z+\omega-\omega^2) - (1-\omega)(1-\omega^2) = 0$

$z^2 = 0$

Only one solution.

48. The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *such that* $\theta \neq \frac{n\pi}{5}$ for n = 0,±1,±2 and tan θ = cot 5θ as well as sin 2θ = cos 4θ is

Ans: 3

Sol. $\tan\theta = \cot 5\theta$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\cos 5\theta}{\sin\theta} \quad \Rightarrow \cos 6\theta = 0$

$\Rightarrow 6\theta = (2n+1)\frac{\pi}{2}$

$\Rightarrow \theta = (2n+1)\frac{\pi}{12} n \in I$

$\Rightarrow \theta = -\frac{5\pi}{12}, -\frac{\pi}{4}, -\frac{\pi}{12}, \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}$ ………. (1)

$\sin 2\theta = \cos 4\theta$

$\Rightarrow \quad \sin 2\theta = 1 - 2\sin^2 2\theta$

$\Rightarrow \quad 2\sin^2 2\theta + \sin 2\theta - 1 = 0$

$\Rightarrow \ \sin 2\theta = -1, \frac{1}{2}$

$\Rightarrow \ 2\theta = (4m-1)\frac{\pi}{2}, p\pi + (-1)^p \frac{\pi}{6}$

$\Rightarrow \ q = (4m-1)\frac{\pi}{4}, \frac{p\pi}{2} + (-1)^p \frac{\pi}{12}; m, p \in I$

$\Rightarrow \ \theta = -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$ ……….. (2)

From (1) and (2)

$$\theta = \left\{-\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}\right\}$$

Number of solution.

49. For any real number x, let [x] denote the largest integer less than or equal to x . Let f be a real valued function defined on the interval [-10,10] by

f(x)= $\begin{cases} x-[x] & if\ [x] is\ odd, \\ 1+[x]-x & if\ [x] is\ even \end{cases}$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\, dx\ is$

Ans: 4

Sol. $f(x) = \begin{cases} \{x\}, 2n-1 \le x < 2n \\ 1-\{x\}, 2n \le x < 2n+1 \end{cases}$

Clearly f(x) is a periodic function with period = 2

Hence f(x) cos πx is also periodic with period = 2

$$\frac{\pi^9}{10}\int_{10}^{10} f(x)\cos(\pi x)\,dx = \pi^2 \int_0^2 f(x)\cos(\pi x)\,dx = \pi^2\int_0^1((1-\{x\}) + \{-x\}\cos(\pi x)\,dx$$

$$= 2\pi^2 \int_0^1 (-x\cos \pi x)\,dx = -2\pi^2\left[\frac{x\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\right]_0^1 = -2\pi^2\left(-\frac{2}{\pi^2}\right) = 4$$

50. If the distance between the plane Ax-2y+z =d and the containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then |d| is

Ans: 6

Sol. Equation of plane is $\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

$x - 2y + z = 0$ …….. (1)

$Ax - 2y + z = d$ …….. (2)

$Compare\ \frac{A}{1} - \frac{-2}{-2} - \frac{1}{1} \Rightarrow A - 1$

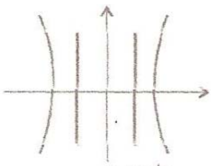
Distance between planes is $\left|\frac{d}{\sqrt{1+1+4}}\right| = \sqrt{6}$

$\Rightarrow |d| = 6.$

51. The line 2x + y =1 is tangent to the hyperbola$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis , then the eccentricity of the hyperbola is

Ans: 2

Sol.



$y = -2x + 1$

$0 = -\left(-\frac{2a}{e} + 1\right)$

$\Rightarrow \frac{a}{e} = \frac{1}{2}$

$e = 2a$

$c^2 = a^2m^2 - b^2$

$\Rightarrow 1 = 4a^2 - b^2$

$\Rightarrow 1 + b^2 - 4a^2 = 0$

$e^2 = 1 + \frac{b^2}{a^2}$

$= 1 + \frac{(4a^2-1)}{a^2}$

$e^2 = 1 + 4 - \frac{1}{a^2}$

$e^2 = 5 - \frac{4}{e^2}$

$\Rightarrow e^4 - 5e^2 + 4 = 0$

$\Rightarrow (e^2-1)(e^2-4) = 0$

$e^2 - 1 \neq 0 \quad e = 2$

52. let S_k, k= 0,1,2,…………….100, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then the value of $\frac{100^2}{100!} + \sum_{k=1}^{100} |k^2 - 3k + 1|S_k|$ is

Ans: 4

Sol. $\sum_{K=1}^{100} |(k^2 - 3k + 1)S_k|$

$\sum_{K=1}^{100} \left(\frac{k-1}{(k-3)!} - \frac{k-1+1}{(k-1)!}\right)$

$\sum_{K=1}^{100} \frac{1}{(k-3)!} + \frac{1}{(k-2)!} - \frac{1}{(k-2)!} - \frac{1}{(k-1)!}$

$\sum_{K=1}^{100} \left(\frac{1}{(k-3)!} - \frac{1}{(k-1)!}\right)$

$S = \left(1 - \frac{1}{2!}\right) + \left(\frac{1}{1!} - \frac{1}{3!}\right) + \left(\frac{1}{2!} - \frac{1}{4!}\right) + \left(\frac{1}{3!} - \frac{1}{5!}\right) + \left(\frac{1}{4!} - \frac{1}{6!}\right) + \cdots \left(\frac{1}{94!} - \frac{1}{96!}\right)$
$+ \left(\frac{1}{95!} - \frac{1}{97!}\right) + \left(\frac{1}{96!} - \frac{1}{98!}\right) + \left(\frac{1}{97!} - \frac{1}{99!}\right) = 2 - \frac{1}{98!} - \frac{1}{99!}$

$\therefore \quad E = \frac{100^2}{100!} + 2 - \frac{1}{98!} - \frac{1}{99.98!}$

$= \frac{100^2}{100!} + 2 - \frac{100}{99!} = \frac{100^2}{100.99!} + 2 - \frac{100}{99!} = 2$

53. Let f be a real-valued differentiable function on R (the set of all real numbers) such that f(1) = 1. If the y-intercept of the tangent at any point P(x,y) on the curve y = f(x) is equal to the cube of the abscissa of P, then the value of f(-3) is equal to

Ans: 9

Sol. Y – y = m (X – x)

y-intercept (x = 0)

y = y – mx

Given that y – mx = x^3 $\quad \Rightarrow \quad x\frac{dy}{dx} - y = -x^3$

$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = -x^2$

Integrating factor $e^{-\int \frac{1}{x}dx} = \frac{1}{x}$

Solution $y.\frac{1}{x} = \int \frac{1}{x}.(-x^2)dx \quad \Rightarrow f(x) = y = -\frac{x^3}{2} + cx$

$Given\ f(1) = 1 \qquad \Rightarrow c = \frac{3}{2}$

$\therefore \quad f(x) = -\frac{x^3}{2} + \frac{3x}{2} \qquad \Rightarrow f(-3) = 9$

54. If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a} = \frac{\widehat{\imath - 2\jmath}}{\sqrt{5}}$ and $\vec{b} = \frac{\widehat{2\imath + \jmath + 3k}}{\sqrt{14}}$, then the value of $(2\vec{a} + \vec{b}).[(\vec{a}x\vec{b})x(\vec{a} - 2\vec{b})]$ is

Ans: 5

Sol. $\vec{a} = \frac{\hat{\imath} - 2\hat{\jmath}}{\sqrt{5}}, \vec{b} = \frac{2\hat{\imath} + \hat{\jmath} + 3\hat{k}}{\sqrt{14}}$

$|\vec{a}| = 1, |\vec{b}| = 1$

$\vec{a}.\vec{b} = 0$

$(2\vec{a} + \vec{b}).\{(\vec{a} \times \vec{b}) \times (\vec{a} - 2\vec{b})\}$

$= (2\vec{a} + \vec{b}).\{\vec{a}(\vec{a} - 2\vec{b})\vec{b}.(\vec{b}(\vec{a} - 2\vec{b})\vec{a})\}$

$= (2\vec{a} + \vec{b}).\{(\vec{a}\vec{a} - 2\vec{a}\vec{b})\vec{b} - (\vec{b}.\vec{a} - 2\vec{b}.\vec{b})\vec{a}\}$

$= (2\vec{a} + \vec{b}).\{(1 - 0)\vec{b} - (0 + 2)\vec{a}\}$

$= (2\vec{a} + \vec{b}).\{\vec{b} + 2\vec{a}\}$

$= 2(\vec{a}.\vec{b}) + 4(\vec{a}.\vec{a}) + \vec{b}.\vec{b} + 2(\vec{b}.\vec{a}) = 0 + 4 + 1 + 0 = 5$.

55. The number of all possible values of θ, where 0<θ<π, for which the system of equations

(y + z) cos 3θ = (xyz) sin 3θ

x sin3θ = $\frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$

(xyz) sin 3θ = (y+2z) cos 3θ + y sin3θ

Have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0, is$

Ans: 3

Sol. Let xyz = t

$t\sin 3\theta - y\cos 3\theta - z\cos 3\theta = 0$ ……… (1)

$t\sin 3\theta - 2y\sin 3\theta - 2z\cos 3\theta = 0$ ……… (2)

$t\sin 3\theta - y(\cos 3\theta + \sin 3\theta) - 2z\cos 3\theta = 0$ ……… (3)

$y_0.z_0 \neq 0$ hence homogeneous equation has non-trivial solution

$$D = \begin{vmatrix} \sin 3\theta & -\cos 3\theta & -\cos 3\theta \\ \sin 3\theta & -2\cos 3\theta & -2\cos 3\theta \\ \sin 3\theta & -(\cos 3\theta + \sin 3\theta) & -2\cos 3\theta \end{vmatrix} = 0$$

$\Rightarrow \sin 3\theta \cos 3\theta (\sin 3\theta - \cos 3\theta) = 0$

$\Rightarrow \sin 3\theta = 0\ or\ \cos 3\theta = 0\ or\ \tan 3\theta = 1$

Case – 1 $\sin 3\theta = 0$

From equation (2)

z = 0 not possible

Case – 1I $\sin 3\theta = 0, \sin 3\theta \neq 0$

$t.\sin 3\theta = 0 \quad \Rightarrow t = 0 \Rightarrow x = 0$

From equation (2)

y = 0 not possible

Case – 1II $\tan 3\theta = 1$

$\Rightarrow 3\theta = n\pi + \frac{\pi}{4}, n \in I$

$\Rightarrow x, y, z \sin 3\theta = 0 \quad \Rightarrow \theta = \frac{n\pi}{3} + \frac{\pi}{12}, n \in I$

$\Rightarrow x = 0, \sin 3\theta \neq 0 \quad \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}$

Hence 3 solutions.

56. The maximum value of the expression $\frac{1}{sin^2\theta + 3\, sin\theta cos\theta + 5cos^2\theta}$ is

Ans: 2

Sol. $f(\theta) = \frac{1}{sin^2\theta + 3\, sin\theta cos\theta + 5cos^2\theta} = \frac{1}{\frac{1-\cos 2\theta}{2} + \frac{3}{2}\sin 2\theta + \frac{5(1+\cos 2\theta)}{2}}$

$\therefore f(\theta)_{max} = \frac{2}{6-5} = 2$

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